

growth involving the margin of the cornea. The tumor had invaded the wall of the eyeball to such an extent that I deemed it advisable to remove the globe, although vision was almost perfect. Enucleation was accordingly performed, and the disease has not returned.

THE EQUIVALENCE OF CYLINDRICAL AND SPHERO-CYLINDRICAL LENSES.

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RECENT discussion as to the use of cylindrical lenses has revealed that some are disposed to employ two cylindrical surfaces where one would be preferable. In view of this, and because the propositions in question are capable of other applications of practical importance, it seems worth while to record the following demonstrations. These demonstrations apply within the same limits and with the same approximation to accuracy, as the demonstrations and formulæ in common use for spherical and cylindrical lenses. That is, the lenses are supposed to be so thin that they may, without serious error, be regarded as superimposed in the same plane; and their spherical aberration so slight that it may be disregarded; while the pencil of incident rays emanates from a point in, or sufficiently near, the principal axis of the lens system.

PROPOSITION I.—*Equal crossed cylinders are optically equivalent to a spherical lens of the same focal length.*¹ If two equal cylindrical lenses C' and C'' are superimposed with their axes perpendicular to one another, the axial plane of each lens will be perpendicular to the axis of the other. Since a cylindrical lens does not affect the direction of incident rays with refer-

¹ The *axial plane* of a cylindrical lens is the plane which passes through the geometrical axis of the cylindrical surface, and the middle of that surface. The line in which it cuts that surface is called the *axis* of the cylindrical lens. Where cylindrical lenses are superimposed the intersection of their axial planes constitutes the *principal axis* of the lens system. Cylindrical lenses superimposed with their axial planes, and therefore their axes, perpendicular to one another, are called *crossed cylinders*.

ence to a plane perpendicular to its axis, the influence exerted on the incident pencil by each lens with reference to its own axial plane will be the same as though the other lens were absent. C' will cause all the rays of the incident pencil to pierce its axial plane in a focal line (linear focus) parallel to the plane of the superimposed lenses, and distant F' from that plane. In the same way C'' will cause all the rays to pierce its axial plane in a similar focal line parallel to the plane of the superimposed lenses, and distant F'' from that plane. But since $C' = C''$ we have the focal distances for the same incident pencil $F' = F''$. Therefore the two focal lines, being equidistant from the plane of the superimposed lenses and both parallel to that plane, must lie in the same plane; and all the rays piercing both of them must pierce them at their intersection. Hence all the rays of the incident pencil are brought to a single point, just as they would be by a spherical lens. That is, when their axes are perpendicular and $C' = C''$,

$$C' \supset C'' = S.$$

PROPOSITION II.—*Crossed cylindrical lenses of unequal refractive power are optically equivalent to a spherical lens, of the same refractive power as one of the given cylindrical lenses, combined with a cylindrical lens equal in refractive power to the difference between the refractive powers of the given cylindrical lenses.* Suppose C' and C'' to be such unequal crossed cylinders. Now consider C'' as composed of C_1 and C , both of which have their axes coincident with that of C'' ; and C_1 having the same refractive power as C' , then

$$C'' = C_1 \supset C \text{ and}$$

$$C' \supset C'' = C' \supset C_1 \supset C.$$

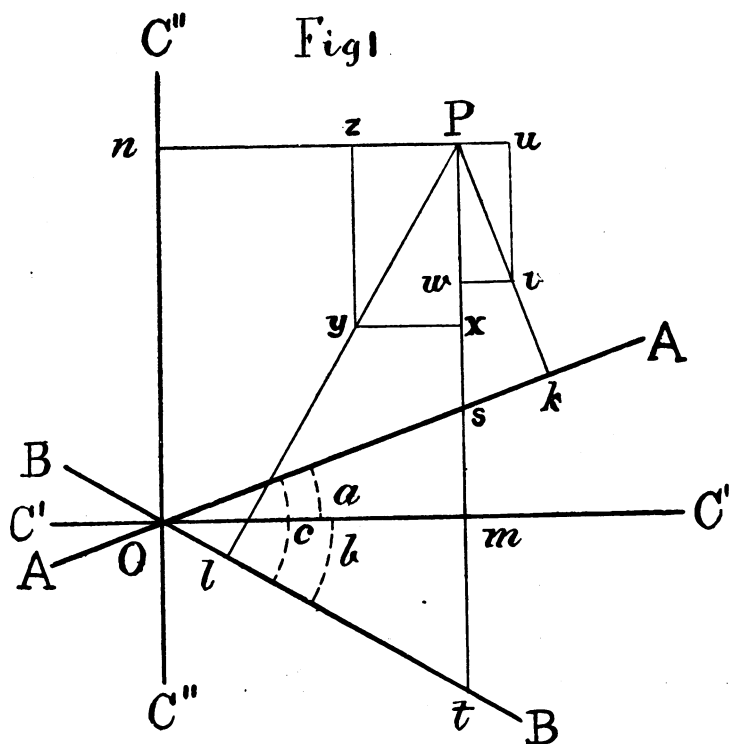
$$\text{But by Prop. I. } C' \supset C_1 = S.$$

$$\text{Hence } C' \supset C'' = S \supset C.$$

The refractive power of C equalling that of C'' minus that of C_1 ($= C'$).

PROPOSITION III.—*Any two superimposed cylindrical lenses with their axes placed at any oblique angle, are optically equivalent to crossed cylinders of certain strengths with their axes placed in certain directions; and therefore to a certain sphero-cylindrical*

lens. Let Fig. I. represent the plane in which are superimposed any two cylindrical lenses, A and B. Let AA and BB making at O any angle c represent the axes of these lenses; and let C'C' and C''C'' represent the axes of the supposed equivalent crossed cylinders, one of them C'C' dividing the angle c into two parts; a adjacent to AA, and b adjacent to BB. From any point P at which an incident ray pierces the plane of the superimposed lenses let fall Pk, Pl, Pm and Pn, respectively perpendicular to AA, BB, C'C' and C''C''; and produce Pm until it cuts BB in t, s being the point in which it cuts AA.



Because their sides are perpendicular, the angle $kPs =$ angle a , and $lPs = b$.

Any ray incident at P is, by the lens A, turned from its course, toward the axial plane of the lens. The force which A

exerts, so to turn the incident ray, may be called r' , which acts in the direction Pk. If now upon Pk we lay off Pv representing r' , and upon Pv as a diagonal complete the parallelogram P u v w, we obtain a parallelogram of forces in which Pv or r' may be considered the resultant of Pu and Pw acting respectively in the directions Pn and Pm; and by the laws governing such forces Pv may for all purposes be replaced by its components Pu and Pw, acting in the directions Pn and Pm. In the same way laying off Py to represent r'' , the force exerted by the lens B upon any ray incident at P, and completing the second parallelogram of forces, it is seen that Py may be replaced by its components Px and Pz also acting in the directions Pm and Pn. Designating the sum of the forces acting in the direction of Pm by R' and of those acting in the direction of Pn by R'' the condition of optical equivalence is that

$$R' + R'' = r' + r'' \quad (0)$$

In considering refraction by lenses it is convenient to designate as positive any force which tends to turn the ray toward the axis of the lens; and as negative any force tending to turn the ray from the axis of the lens. Since Pw and Px are both positive,*

$$R' = Pw + Px$$

But as Pu is negative,

$$R'' = Pz - Pu.$$

Since Pm and Pn are perpendicular to one another, Pw v is a right-angled triangle, so that either of the other sides is equal to the hypotenuse multiplied by the cosine of the adjacent, or sine of the opposite angle.

$$Pw = Pv \times \cos v \quad Pw = r' \cos a$$

$$\text{and } wv = Pu = Pv \times \sin v \quad Pw = r' \sin a.$$

In the same way

$$Px = Py \times \cos x \quad Py = r'' \cos b$$

$$\text{and } xy = Pz = r'' \sin b$$

$$\text{Hence } R' = r' \cos a + r'' \cos b \quad (1)$$

$$\text{and } R'' = r'' \sin b - r' \sin a \quad (2)$$

* I have taken in Fig. I. the case of A and B both convex, and P outside of the angle c; but the same formulæ, with the obviously necessary changes of signs, apply, when either or both of the lenses are concave, or P falls within the angle c.

The refractive power or force (r) of a cylindrical lens at any point may be represented by the distance (d) of that point from the axis of the lens, divided by the focal distance (f) of the lens.

$$\text{Thus } r = \frac{d}{f}$$

Representing by d' , d'' , D' and D'' the respective distances Pk, Pl, Pm and Pn, of P from the axes of A, B, C' and C''; and by f' , f'' , F' and F'' the focal distances of these same lenses; we have

$$r' = \frac{d'}{f'}, \text{ and } r'' = \frac{d''}{f''}$$

Substituting these values of r' and r'' , equations (1) and (2) become

$$R' = \frac{d'}{f'} \cos a + \frac{d''}{f''} \cos b \quad (3)$$

$$R'' = \frac{d''}{f''} \sin b - \frac{d'}{f'} \sin a \quad (4)$$

It remains to be shown that

$$R' = \frac{D'}{F'} \text{ and } R'' = \frac{D''}{F''}$$

for all possible positions of P; and to determine the values of F' and F'' .

In the right-angled triangle Pks, $d' = \text{Pk} = \text{Ps} \times \cos a$.

$$\text{But } \text{Ps} = \text{Pm} - \text{ms} = D' - \text{ms}$$

and in the right-angled triangle Oms;

$$\text{ms} = \text{Om} \times \tan a = D'' \tan a$$

$$\text{Hence } \text{Ps} = D' - D'' \tan a$$

$$\text{and } d' = (D' - D'' \tan a) \cos a$$

Again in the right-angled triangle Plt

$$d'' = \text{Pl} = \text{Pt} \times \cos b$$

$$\text{But } \text{Pt} = \text{Pm} + \text{mt} = D' + \text{mt}$$

and in the right-angled triangle Omt

$$\text{mt} = \text{Om} \times \tan b = D'' \tan b$$

$$\text{Hence } \text{Pt} = D' + D'' \tan b$$

$$\text{and } d'' = (D' + D'' \tan b) \cos b$$

Substituting these values of d' and d'' in equations (3) and (4), we obtain

$$R' = \frac{(D' - D'' \tan a) \cos^2 a}{f'} + \frac{(D' + D'' \tan b) \cos^2 b}{f''}$$

and

$$R'' = \frac{(D' + D'' \tan b) \sin b \cos b}{f''} - \frac{(D' - D'' \tan a) \sin a \cos a}{f''}$$

But $\tan = \frac{\sin}{\cos}$ Substituting and reducing

$$R' = \frac{D' \cos^2 a - D'' \sin a \cos a}{f'} + \frac{D' \cos^2 b + D'' \sin b \cos b}{f''}$$

$$R' = D' \left(\frac{\cos^2 a}{f'} + \frac{\cos^2 b}{f''} \right) + D'' \left(\frac{\sin b \cos b}{f''} - \frac{\sin a \cos a}{f'} \right) \quad (5)$$

and

$$R'' = \frac{D' \sin b \cos b + D'' \sin^2 b}{f''} - \frac{D' \sin a \cos a - D'' \sin^2 a}{f'}$$

$$R'' = D'' \left(\frac{\sin^2 b}{f''} + \frac{\sin^2 a}{f'} \right) + D' \left(\frac{\sin b \cos b}{f''} - \frac{\sin a \cos a}{f'} \right) \quad (6)$$

Equations (5) and (6) being true for all values of a and b , to render R' independent of D'' , and R'' independent of D' , we have only to give a and b such values that

$$\sin 2a : \sin 2b :: f' : f''$$

$$\text{then } \frac{\sin 2a}{f'} = \frac{\sin 2b}{f''}$$

But the sine of double an angle equals double the product of its sine and cosine, so the above becomes

$$\frac{2 \sin a \cos a}{f'} = \frac{2 \sin b \cos b}{f''}$$

$$\text{and } \frac{\sin a \cos a}{f'} - \frac{\sin b \cos b}{f''} = 0$$

The co-efficient of D'' in (5) and D' in (6) being zero, these equations become

$$R' = D' \left(\frac{\cos^2 a}{f'} + \frac{\cos^2 b}{f''} \right) \quad (7)$$

$$R'' = D'' \left(\frac{\sin^2 a}{f'} + \frac{\sin^2 b}{f''} \right) \quad (8)$$

For any given pair of lenses in a given position, the co-efficients of D' in (7) and of D'' in (8) are constant, and may be represented thus :

$$\frac{\cos^2 a}{f'} + \frac{\cos^2 b}{f''} = \frac{1}{F'}$$

$$\frac{\sin^2 a}{f'} + \frac{\sin^2 b}{f''} = \frac{1}{F''}$$

from which F' and F'' the focal distances of C' and C'' might be easily calculated; and making equations (7) and (8) take the form

$$R' = \frac{D'}{F'} \text{ and } R'' = \frac{D''}{F''}$$

It is evident that no other values of a and b or F' and F'' will satisfy the above equations; hence C' and C'' in the position assumed, are the only crossed cylinders that will exactly replace A and B .

$$\begin{aligned} \text{Since } A \supset B &= C' \supset C'' \\ \text{and (Prop. II.) } C' \supset C'' &= S \supset C \\ A \supset B &= S \supset C \end{aligned}$$

PROPOSITION IV.—*Any number of superimposed cylindrical lenses, with their axes placed in any directions, may be optically replaced by a single sphero-cylindrical lens.*

Suppose $C' \supset C'' \supset C''' \dots C^n$ to be a series of superimposed cylindrical lenses. By Prop. III. we may get

$$S_1 \supset C_1 = C' \supset C''$$

Then combining C_1 with C''' we have

$$S_{11} \supset C_{11} = C_1 \supset C'''$$

adding S_1 we have

$$S_1 + S_{11} \supset C_{11} = S_1 \supset C_1 \supset C''' = C' \supset C'' \supset C'''$$

continuing and calling the cylindrical finally obtained C we have

$$S_1 + S_{11} \dots S_{n-1} \supset C = C' \supset C'' \supset C''' \dots C^n$$

$$\text{and making } S = S_1 + S_{11} \dots + S_{n-1}$$

$$\text{we have } C' \supset C'' \supset C''' \dots C^n = S \supset C$$

In Proposition III. the values of a and b , and F' and F'' , might be calculated directly from the formulæ given; but an easier way of obtaining them is developed as follows: Designating by $\frac{1}{F}$ the refractive power of C , we have

$$\begin{aligned}\frac{1}{F} &= \frac{1}{F'} - \frac{1}{F''} \\ &= \frac{\cos^2 a}{f'} + \frac{\cos^2 b}{f''} - \frac{\sin^2 a}{f'} - \frac{\sin^2 b}{f''} \\ &= \frac{\cos^2 a - \sin^2 a}{f'} + \frac{\cos^2 b - \sin^2 b}{f''}\end{aligned}$$

$$\begin{aligned}\text{But } \cos^2 a - \sin^2 a &= \cos 2a \\ \text{and } \cos^2 b - \sin^2 b &= \cos 2b\end{aligned}$$

$$\text{Hence } \frac{1}{F} = \frac{\cos 2a}{f'} + \frac{\cos 2b}{f''}$$

$$\begin{aligned}\text{But if } \frac{\sin 2a}{f'} &= \frac{\sin 2b}{f''} \\ f'' &= \frac{f' \sin 2b}{\sin 2a}\end{aligned}$$

$$\frac{1}{F} = \frac{\cos 2a}{f'} + \frac{\cos 2b \sin 2a}{f' \sin 2b}$$

multiplying by $\sin 2b$ we get

$$\frac{\sin 2b}{F} = \frac{\cos 2a \sin 2b + \cos 2b \sin 2a}{f'}$$

But referring again to trigonometry

$$\cos 2a \sin 2b + \cos 2b \sin 2a = \sin (2a + 2b) = \sin 2c$$

$$\text{hence } \frac{\sin 2b}{F} = \frac{\sin 2c}{f'}$$

$$\sin 2b : \sin 2c :: \frac{1}{f'} : \frac{1}{F}$$

$$\sin 2b : \frac{1}{f'} :: \sin 2c : \frac{1}{F}$$

$$\text{But } \sin 2a : \frac{1}{f''} :: \sin 2b : \frac{1}{f'}$$

$$\text{Hence } \frac{1}{f''} : \sin 2a :: \frac{1}{f'} : \sin 2b :: \frac{1}{F} : \sin 2c \text{ (A)}$$

To obtain a , b , and $\frac{1}{F}$ by construction, in general (as Stokes did in the particular case of his lens): From any point O (Fig. 2) draw O L to represent $\frac{1}{f'}$, and O M to represent $\frac{1}{f''}$, making the angle L O M = $2c$, complete the parallelogram and draw the diagonal O N.

$$A \subset A^r \subset B \subset B^r = 2 S \subset C \subset C^r$$

But by Prop. I. $A \subset A^r = A^s$

A^s being a spherical lens of the same refractive power as A .

In the same way $B \subset B^r = B^s$

and $C \subset C^r = C^s$

But the combination of any number of spherical lenses is equivalent to a spherical lens, the refractive power of which is equal to the sum of their refractive powers.

$$\text{Hence } \frac{1}{f'} + \frac{1}{f''} = \frac{2}{F''} + \frac{1}{F}$$

$$\text{or } \frac{1}{F''} = \frac{1}{2} \left(\frac{1}{f'} + \frac{1}{f''} - \frac{1}{F} \right)$$

To find the sphero-cylindrical equivalent of any two superimposed cylindrical lenses :

1st.—Construct a parallelogram two adjacent sides of which represent the refractive powers of the given lenses, while their included angle equals twice the angle made by the axes of the lenses, if both are convex or concave; and the supplement of twice that angle, if one be convex and the other concave. The diagonal cutting this included angle, will correspondingly represent the refractive power, and direction of the axis, of the cylindrical portion of the equivalent.

2d.—From half the sum of the refractive powers of the given lenses subtract half the refractive power of the cylindrical portion of the equivalent; the remainder will be the refractive power of the spherical portion of the equivalent.

DISCUSSION.

DR. HAY.—I have been much interested in the demonstration, which is similar to that which was given to me by Dr. Oliver, now Professor Oliver, of Cornell University. In Donders' book is given the formula for determining the equivalent of Stokes' lens, which consists of two cylinders of equal power, one convex and the other concave, rotated so that their axes make successive angles. I tried to work the formula out, but did not succeed. I mentioned the matter to Dr. Oliver, and he went to the black-board and tried to work it out. The following day he sent me a demonstration, using

a figure similar to that shown by Dr. Jackson. He did not tell me how he did certain parts of it, but I subsequently worked it out in a way a little different from that given to-day. This method may be applied either where the two lenses which you want to combine are both convex or both concave, or where one is convex and the other concave.

There is one other point, on which I would express myself a little differently from the speaker. I should not say that they are perfect equivalents. I should say that the resultants are equivalent for very small pencils. The smaller the pencil, the less error is there in the statement of the equivalence. You might decompose the effect from one cylinder, say C' , into two components, one with reference to C'' and another with reference to C_1 ; these two components being approximately correct when the pencil is small. It is true for most formulæ that they hold only for small pencils.

DR. JACKSON.—In the beginning of my paper, I stated that this demonstration was only true within the same limits as apply to the optical formulæ in common use; thin lenses, small apertures, and pencils of light originating from points close to the axis of the lens system.

With reference to the position of the axis. If we regard every cylinder as having an axis of greatest and of least refraction, then in a convex cylinder its axis of greatest refraction will be the axis of the cylinder, and its axis of least refraction will be perpendicular to this; in a concave cylinder, the axis of greatest refraction would not be where the refraction of the cylinder is really the greatest, but where there is no refraction at all, that is to say, — R is less than O . In the one case it is where $+R$ is at its maximum, and in the other case where $R = O$. The general formula will apply by observing the proper signs; and the axis of greatest refraction for the equivalent cylinder will always lie between the axes of greatest refraction of the given cylinders.

DR. DENNETT.—This demonstration was said to be correct for very small pencils of light and thin lenses. This calls to mind the fact that these formulæ are not always correct for the lenses which patients require. Dr. Loring has had some experience in prescribing crossed cylinders for cataract. He gave a satisfactory demonstration to the New York Ophthalmological Society, that the images formed by these crossed cylinders were clearer than the images formed by the equivalent sphero-cylindrical lenses. The difference was so marked that patients insisted upon having their crossed cylinders. And it has long been known that watch makers' magnifying

glasses and some of the best of reading glasses were made of crossed cylinders, on account of popular impression among those using them that they were better. A hurried examination gives one the idea that the field is larger, and that there is not so much distortion at the edges as is seen through spherical glasses of the same size and strength.

DR. LITTLE.—With reference to the practical side of this question, I went over my case books to ascertain how often I had given crossed cylinders in mixed astigmatism, and how often I had given the equivalent sphero-cylinder. I found that I had given sphero-cylinders in thirty-three cases, and had given crossed cylinders with the angles not at right angles in thirty-five. The reason that I gave crossed cylinders in preference to their equivalent sphero-cylinder was because the patients preferred them. In placing a sphero-cylinder which corrects mixed astigmatism before a patient, you have to be careful that the glass is accurately centred. I have in mind a case which I corrected seven years ago, using a sphero-cylinder for one eye and crossed cylinders for the other. Where the crossed cylinders were used, there was a slight corneal opacity, and better vision was obtained. There is a practical advantage, but I am not quite convinced why it is.

A case recently presented itself with cataract in one eye and high astigmatism in the other. A minus cylinder at 90° , brought vision up to $\frac{20}{80}$; a + cylinder added at 45° , gave $V = \frac{20}{20}$. I am not prepared to mathematically demonstrate what I believe, but I know that patients will many times select crossed cylinders in preference to their equivalent in sphero-cylinders.

THE AMBLYOPIA OF SQUINTING EYES: IS IT A DETERMINING CAUSE, OR A CONSEQUENCE OF THE SQUINT?

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THE question whether the amblyopia which exists in the squinting eye in so large a proportion of cases of concomitant convergent strabismus is a consequence of the squint, or whether it antedates it, and is an important factor in deter-